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芝浦工業大学大学院工学マネジメント研究科

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LIBOR Market Model under the Real-world Measure, and Real-world Simulation

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Abstract

This paper consists of two parts. The first part aims to construct a LIBOR market model under the real-world measure (LMRW) according to the Jamshidian framework. Then, LIBOR rates, bond prices and a state price deflator are explicitly described under the LMRW. Additionally, the LIBOR market model under the spot LIBOR measure is induced from the LMRW. The second part aims to estimate the market price of risk, as well as to investigate the fundamental properties of real-world simulations. Then, the following subjects are theoretically investigated:
(1) a method for determining the number of factors for real-world simulations, (2) the properties of real-world simulations and the function of the market price of risk, and (3) the value of the market price of risk in connection with sample data. Numerical examples demonstrate our results.

Keywords : LIBOR market model, term structure model, market price of risk, real-world simulation, principal component analysis.

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1 Introduction

There are two main objectives for studying an interest rate term structure model. The first is to study term structure dynamics from an econometrics point of view, where an affine term structure model (Duffie and Kan [4]) is used under a real-world (or historical) measure. The second is to price and hedge interest rate derivatives, where an arbitrage-free term structure model is used under a risk-neutral measure. In practice, the standard model for the latter purpose is the LIBOR market model introduced by Miltersen et al. [9], Brace et al. [1] (hereafter BGM), Musiela and Rutkowski [10], and Jamshidian [6]. This model was developed for use in derivative pricing.

Furthermore the term structure model can be applied to risk assessment for a portfolio subject to interest rate risk. In this case, the model should be used under the real-world measure, for details see Rebonato et al. [12]. Also, it is desirable to use an arbitrage-free term structure model for this objective.

Hence, studying real-world simulations employing the LIBOR market model is useful because of its prevalence and positive rates of forward LIBOR. In regard to this issue, Norman [11] constructs the BGM model under a real-world measure, and develops a real-world evolution of the forward LIBOR curve. Along this line, we study real-world simulation using the LIBOR market model.

This paper consists of two parts: the first aims to construct the LIBOR market model under the real-world measure (LMRW) in a rigorous manner. We work within the Jamshidian framework [6] focusing on the relation between the real world and the risk-neutral world, since Jamshidinan [6] gives the only model that is rigorously constructed in connection to the real-world measure. In Section 2, we recall the framework of the LIBOR market model under the spot LIBOR measure (LMSP) according to [6]. In Section 3, we define the LMRW and show its existence. In Section 4, from the LMRW we induce the LIBOR market model under the spot LIBOR measure and under the forward measure by examining changes in the measures.

The second part aims to study fundamental properties of the market price of risk and simulations using the LMRW. To that end, it is essential to estimate the market price of risk in the LMRW. In the field of econometrics, many previous studies estimate the market price of risk in an affine model. For example, Stanton[14] use regression analysis on historical yield data to estimate the market price of risk in an affine model framework. Dempster et al. [3] use a Kalman filter to estimate the market price of risk in the three-factor Vasicek model [15]. Under the LIBOR market model, Norman [11] gives a method for calculating the market price of risk in a LIBOR market model dealing with mean reversion. These approaches can be executed by using an econometric software package to numerically evaluate the market price of risk, but it is difficult to use them to investigate theoretically the properties of real-world simulations.

This paper therefore exactly estimates the market price of risk, and theoretically studies fundamental properties of real-world simulations. Section 5 introduces an exact expression for the market price of risk by solving a least square problem for a lag regression model, making such studies possible and leading to several interesting results. In this approach, the term "MPR score" is defined to give a simple expression for the market price of risk. We next investigate a drift term of real-world simulation in connection with historical trends. Results show that real-world simulations are similar to an empirical model with historical trends of the forward LIBOR and historical volatility.

Section 6 explores properties of real-world simulations in the LMRW for practical use. The most basic problem is determining the number of factors for the simulation, which we show is determined by principal component analysis and the MPR score. Next, properties of the market price of risk are investigated in connection with LIBOR historical trends. Many studies have attempted to explain the market price of risk using state variables, but our results explain the market price of risk using changes in the historical forward LIBOR curve rather than the curve state.

Finally, we give a numerical procedure for real-world simulations, and in Section 7 demonstrate our results using numerical examples.

2 Bond price process and LIBOR market model

We recall the LIBOR market model according to Jamshidian [6] (cf. Schoenmakers [13]). This section is split into two subsections. The first subsection introduces some basic concepts of the bond price system, and the second one describes the LMSP.

2.1 Bond price process

Let $T > 0$ be a time horizon, and let $(\Omega, F_t, \mathbf{P}),\ t \in [0,T]$ be a probability space, where F_t is the augmented filtration, and $\mathbf{P}$ is the original measure, which we call the *real-world measure*. Let $\mathcal{E}$ be a set of continuous semimartingales on $[0,T]$, and $\mathcal{E}_+ = \{X \in \mathcal{E} \mid X > 0\}$, $\mathcal{E}^n = \{X \mid X = (X_1, \cdots, X_n), X_i \in \mathcal{E}\}$, and so on.

Let $\mathbf{Q}$ be a measure equivalent to $\mathbf{P}$. We denote the F_t-conditional expectation operator by E_t. Letting $M = E_t[d\mathbf{Q}/d\mathbf{P}]$, $M(0) = 1,\ M > 0$ and M is a $\mathbf{P}$-martingale. The following proposition is well known and important for the change of numeraire.

Proposition 2.1 *$X \in \mathcal{E}$ is a $\mathbf{Q}$-martingale if and only if XM is a $\mathbf{P}$-martingale.*

If the covariance process of $X, Y \in \mathcal{E}$ is denoted by $\langle X, Y\rangle$, then the following proposition holds.

Proposition 2.2 *Let $\mathbf{P}$, $\mathbf{Q}$ and M be same as before. If X is a $\mathbf{P}$-Brownian motion, then $Y = X - \langle X, \log M\rangle$ is a $\mathbf{Q}$-Brownian motion.*

$B \in \mathcal{E}^n$ is said to be *arbitrage-free* if there exists $\xi \in \mathcal{E}_+$ with $\xi(0) = 1$, such that ξB_i are $\mathbf{P}$-martingales for all i. ξ is called the *state price deflator*. Let $B \in \mathcal{E}^n$ be arbitrage-free, and assume that $B_i > 0$ on $[0,T]$ for some i. Then, $\xi B_i/B_i(0)$ are positive $\mathbf{P}$-martingales with $\xi B_i/B_i(0) = 1$. The B_i*numeraire measure* $\mathbf{P}_i$ is defined by a measure equivalent to $\mathbf{P}$ such that

$$d\mathbf{P}_i/d\mathbf{P} = \xi(T)B_i(T)/B_i(0).$$

More generally, for $C \in \mathcal{E}_+$ such that ξC is a $\mathbf{P}$-martingale, the *Cnumeraire measure* $\mathbf{P}^C$ is a measure equivalent to $\mathbf{P}$ defined by

$$d\mathbf{P}^C/d\mathbf{P} = \xi(T)C(T)/C(0).$$

Let $\theta = (\theta_1, \cdots, \theta_n)$ be a vector process such that each θ_i is a predictable B_i-integrable process. The pair (θ, B) is called a *self-financing trading strategy (SFTS)* if

$$\theta \cdot B = \theta(0) \cdot B(0) + \int_0^t \theta \cdot dB$$

for all $0 \le t \le T$.

If we assume that B_i and ξ are Ito processes with respect to a d-dimensional $\mathbf{P}$-Brownian motion Z, then

$$\begin{aligned} B_i(t) &= B_i(0) \exp\left\{ \int_0^t \left(\mu_i - \frac{|\sigma_i|^2}{2} \right) ds + \int_0^t \sigma_i \cdot dZ \right\} \\ \xi(t) &= \exp\left\{ \int_0^t \left(-r - \frac{|\varphi|^2}{2} \right) ds - \int_0^t \varphi \cdot dZ \right\}, \end{aligned}$$

where $\mu_i, r : \Omega \times [0,T] \to \mathbf{R}$ and $\sigma_i, \varphi : \Omega \times [0,T] \to \mathbf{R^d}$ are predictable processes satisfying

$$\int_0^T |\mu_i| ds < \infty, \qquad \int_0^T |\sigma_i|^2 ds < \infty, \qquad \int_0^T |r| ds < \infty, \qquad \int_0^T |\varphi|^2 ds < \infty \ .$$

From Ito's lemma we have the following equations

$$\frac{dB_i}{B_i} = \mu_i dt + \sigma_i \cdot dZ, \qquad \frac{d\xi}{\xi} = -r dt - \varphi \cdot dZ. \tag{2.1}$$

If there exist φ and r such that μ_i and σ_i satisfy

$$\mu_i - r = \sigma_i \cdot \varphi, \tag{2.2}$$

for every i, B is arbitrage-free. Also, (2.2) implies

$$\mu_i - \mu_j = (\sigma_i - \sigma_j) \cdot \varphi. \tag{2.3}$$

We call φ the *market price of risk*, and r the *implied instantaneous interest rate*. Note that the market price of risk is not unique since we do not assume that the market is complete.

2.2 LIBOR market model under the spot LIBOR measure

Let $0 < T_1 < \cdots < T_n = T$ be a sequence of time, which we call a *tenor structure*. We set $\delta_i = T_{i+1} - T_i$ for all i, and define a left continuous function $m(t)$ to be the unique integer such that

$$T_{m(t)-1} < t \le T_{m(t)}.$$

We assume that B_i are the prices of a zero-coupon bond with face value 1 at maturity T_i. The forward LIBOR process L_i is defined by

$$1 + \delta_i L_i = B_i / B_{i+1}. \tag{2.4}$$

Naturally $L_i(t)$ means the forward rate observed at t over the period $[T_i, T_{i+1}]$.

We set

$$\theta_i^* = \frac{1_{\{T_{m(t)-1}<t\le T_{m(t)}\}}}{B_1(0)} \prod_{j=1}^{m(t)-1} \frac{B_j(T_j)}{B_{j+1}(T_j)}, \quad 0 < t \le T, \tag{2.5}$$

and $\theta_1^*(0) = 1/B_1(0),\ \theta_i^*(0) = 0,\ 1 < i \le n$. Then (θ^*, B) is an STFS, and B^* is defined by

$$B^* = \theta^* \cdot B = \frac{B_{m(t)}(t)}{B_1(0)} \prod_{j=1}^{m(t)-1} (1 + \delta_j L_j(T_j)). \tag{2.6}$$

This trading strategy (θ^*, B) is feasible by trading at $T_i,\ 0 < i \le n$, thus we may take B^* as a numeraire.

In particular, we consider a case where B is arbitrage-free and both B_i and ξ are Ito processes. Then we have

$$dB^*/B^* = dB_{m(t)}/B_{m(t)} = \mu_{m(t)}dt + \sigma_{m(t)} \cdot dZ \tag{2.7}$$

The spot LIBOR measure $\mathbf{P}^*$ is defined as the B^* numeraire measure by $d\mathbf{P}^*/d\mathbf{P} = \xi(T)B^*(T)$. It follows that

$$d\langle Z,\ \log(\xi B^*)\rangle = -\varphi + \sigma_{m(t)}$$

We set $Z^* \equiv Z + \int_0^t (\varphi - \sigma_{m(s)})dt$. From Proposition 2.2, Z^* is a $\mathbf{P}^*$-Brownian motion.

From (2.1) and (2.4) we have

$$\begin{aligned} dL_i &= \frac{(1+\delta_i L_i)(\sigma_i - \sigma_{i+1})}{\delta_i} \cdot \left\{ \left(\sum_{j=m(t)}^{i} (\sigma_j - \sigma_{j+1}) + \varphi - \sigma_{m(s)} \right) dt + dZ \right\} \\ &= \frac{(1+\delta_i L_i)(\sigma_i - \sigma_{i+1})}{\delta_i} \cdot \{(\sigma_{m(t)} - \sigma_{i+1})dt + dZ^*\} \end{aligned} \tag{2.8}$$

(for details, see [6], p. 316). We set

$$\lambda_i = \frac{1+\delta_i L_i}{\delta_i L_i}(\sigma_i - \sigma_{i+1}) \tag{2.9}$$

for every i. Then the equation for L_i becomes

$$\frac{dL_i}{L_i} = \sum_{j=m(t)}^{i} \frac{\delta_j L_j \lambda_i \cdot \lambda_j}{1+\delta_j L_j} dt + \lambda_i \cdot dZ^* \tag{2.10}$$

for every i. Note that λ_i are predictable, but not deterministic. Here we give the definition of the LIBOR market model according to Jamshidian[6] as follows.

Definition 2.2 $B \in \mathcal{E}^n$ is called *a LIBOR market model* if B is arbitrage-free such that $L_i > 0$ and $< \log L_i, \log L_j >$ are deterministic for all i, j.

Therefore the LIBOR process of (2.10) does not give a LIBOR market model, while λ_i are not deterministic.

Conversely, we assume that λ is deterministic, namely $\lambda_i = \lambda_i(t)$. Consider the following equation.

$$\frac{dL_i}{L_i} = \sum_{j=m(t)}^{i} \frac{\delta_j L_j \lambda_i(t) \cdot \lambda_j(t)}{1+\delta_j L_j} dt + \lambda_i(t) \cdot dZ^Q, \tag{2.11}$$

where Z^Q is a Brownian motion under an equivalent measure $\mathbf{Q}$. This equation admits a unique positive solution for an arbitrary initial condition $L_i(0) > 0,\ 0 \le i \le n-1$.

Let B^* be an arbitrary process in $\mathcal{E}_+$ such that the following equality holds at every tenor date T_i,

$$B^*(T_i) = \frac{1}{B_1(0)} \prod_{j=1}^{i-1} (1 + \delta_j L_j(T_j)), \quad i \le n-1. \tag{2.12}$$

The most important example of B^* is a process with null volatility obtained by log linear interpolation such that

$$\log B^*(t) = \log B^*(T_{i-1}) + \frac{\log(1+\delta_{i-1}L_{i-1})}{\delta_{i-1}}(t - T_{i-1}), \qquad t \in (T_{i-1}, T_i], \tag{2.13}$$

which shall be used in Section 4.1. Accordingly B_i is defined by

$$\frac{B_i(t)}{B^*(t)} = \frac{B_1(0)}{\Pi_{j=1}^{m(t)-1}(1+\delta_j L_j(T_j))\ \Pi_{j=m(t)}^{i-1}(1+\delta_j L_j(t))}$$

for $i,\ 1 \le i \le n$. Obviously $B_i(T_i) = 1$ and (2.4) is satisfied. As a result, it is shown in [6](p.317) that B_i/B^* is a $\mathbf{Q}$-martingale for every i. We define ξ by

$$\xi(t) = E_t[d\mathbf{Q}/d\mathbf{P}]/B^*.$$

It is obvious that ξB^* is a $\mathbf{P}$-martingale; thus ξ is the state price deflator.

Defining the spot LIBOR measure (B^* numeraire measure) $\mathbf{P}^*$ by $d\mathbf{P}^*/d\mathbf{P} = \xi(T)B^*(T)$, we have $\mathbf{P}^* = \mathbf{Q}$ a.e. From $\xi B_i = \xi B^*(B_i/B^*)$ and Proposition 2.1, ξB_i are $\mathbf{P}$-martingales, hence B is arbitrage-free. Since L_i are expressed under the spot LIBOR measure $\mathbf{P}^*$, B is called a LIBOR market model under the spot LIBOR measure (LMSP).

Next, we consider a European option whose payoff at T_i is given by C_{T_i}. The price C_t of this option at t is given by

$$C_t = B^*(t) E_t^{\mathbf{P}^*}\left[C_{T_i}/B^*(T_i)\right],$$

where we denote the conditional expectation under $\mathbf{P}^*$ by $E_t^{\mathbf{P}^*}$. At $t = 0$, it follows that

$$C_0 = E^{\mathbf{P}^*}\left[C_{T_i}/B^*(T_i)\right].$$

Therefore when we deal with a derivative whose cash flow occurs at T_i, the option price is not affected by the non-uniqueness of B.

Note that the equivalent measure $\mathbf{Q}$ is not uniquely determined since the Jamshidian model does not assume that the market is complete. However, the option price is independent of the choice of $\mathbf{Q}$ if the payoff is a function of $L(T_1), \cdots, L(T_n)$. For details, see [6].

3 LIBOR market model under the real-world measure

Let $0 < T_1 < \cdots < T_n = T$ be the same as in Section 2.2, and let B_i of $B \in \mathcal{E}^n$ be the price of a zero-coupon bond with face value 1 at maturity T_i. We define the LMRW as follows.

Definition 3.1 B is an *LMRW* if the following two conditions are satisfied.
1) B is arbitrage-free such that $L_i > 0$ and $< \log L_i, \log L_j >$ is deterministic for all i, j.
2) $B \in \mathcal{E}^n_+$ and the state price deflator $\xi \in \mathcal{E}_+$ are Ito processes that are specified under $\mathbf{P}$.

Here note the following. The definition of the LIBOR market model does not require that B_i be Ito processes; hence the model refers neither to the market price nor to the real-world measure. Indeed the LIBOR process with a given volatility structure is constructed under some risk-neutral measure, and B is not expressed as an Ito process.

We define a predictable process $\bar{\mu}$ by $\bar{\mu}(t) = \bar{\mu}(T_{m(t)})$, where $\bar{\mu}(T_i)$ is defined for each i by

$$\bar{\mu}(T_i) = \log\{1 + \delta_{i-1} L_{i-1}(T_{i-1})\}/\delta_{i-1}. \tag{3.1}$$

Note that $\bar{\mu}(t)$ does not change during $(T_{i-1}, T_i]$, $i = 1, \cdots, n$. We shall see in the proof of the next proposition that $\bar{\mu}(t)$ represents the constant yield for the shortest maturity bond, and that simultaneously $\bar{\mu}$ represents the implied instantaneous interest rate. The next proposition shows the existence of the LMRW.

Proposition 3.1 *Let $\varphi : \Omega \times [0, T] \to \mathbf{R^d}$ be an arbitrary predictable process with $\int_0^T |\varphi|^2 ds < \infty$, and let $\lambda_i : [0, T] \to \mathbf{R^d}$ be an arbitrary deterministic process for all i. Let L_i be the solution to the following equation with initial $L_i(0) > 0$,*

$$\frac{dL_i}{L_i} = \left\{ \lambda_i \cdot \sum_{j=m(t)}^{i} \beta_j + \lambda_i \cdot \varphi \right\} dt + \lambda_i(t) \cdot dZ, \qquad i = 1, \cdots, n, \tag{3.2}$$

where $\beta_j = \lambda_j \delta_j L_j/(1+\delta_j L_j)$. Assume that B_i are Ito processes with initial values $B_i(0) = \prod_{j=0}^{i-1}(1+\delta_j L_j(0))^{-1}$ such that

$$\frac{dB_i}{B_i} = \left\{ \bar{\mu} - \sum_{j=m(t)}^{i-1} \beta_j \cdot \varphi \right\} dt - \sum_{j=m(t)}^{i-1} \beta_j \cdot dZ \tag{3.3}$$

Then B is an LMRW, and φ is a market price of risk.

Proof Since $L_i > 0$, $\bar{\mu}$ is positive and well-defined. We set$B_0(0) = 1$. At $t = 0$, we may assume that

$$1 + \delta_i L_i(0) = B_i(0)/B_{i+1}(0), \qquad i = 0, \cdots, n-1. \tag{3.4}$$

First we shall show that, for all i,

$$1 + \delta_i L_i = B_i/B_{i+1}. \tag{3.5}$$

Equation (3.3) admits a unique positive solution

$$B_i(t) = B_i(0)\exp\left\{\int_0^t \left(\bar{\mu} - \sum_{j=m(s)}^{i-1} \beta_j \cdot \varphi - |\sum_{j=m(s)}^{i-1} \beta_j|^2/2\right) ds - \int_0^t \sum_{j=m(s)}^{i-1} \beta_j \cdot dZ\right\}. \tag{3.6}$$

It follows that

$$\begin{aligned}\frac{B_i(t)}{B_{i+1}(t)} &= \frac{B_i(0)}{B_{i+1}(0)}\exp\left\{\int_0^t \left(\beta_i\cdot\varphi + |\sum_{j=m(s)}^{i}\beta_j|^2/2 - |\sum_{j=m(s)}^{i-1}\beta_j|^2/2\right) ds + \int_0^t \beta_i \cdot dZ\right\} \\ &= \frac{B_i(0)}{B_{i+1}(0)}\exp\left\{\int_0^t \left(\beta_i\cdot\varphi + \sum_{j=m(s)}^{i}\beta_j\cdot\beta_i - |\beta_i|^2/2\right) ds + \int_0^t \beta_i\cdot dZ\right\} \end{aligned} \tag{3.7}$$

Equation (3.2) implies

$$\begin{aligned}\frac{d\delta_i L_i}{1+\delta_i L_i} &= \frac{\delta_i L_i}{1+\delta_i L_i}\left\{\left(\lambda_i \cdot \sum_{j=m(s)}^{i}\beta_j + \lambda_i\cdot\varphi\right) dt + \lambda_i \cdot dZ\right\} \\ &= \left(\sum_{j=m(s)}^{i}\beta_j\cdot\beta_i + \beta_i\cdot\varphi\right) dt + \beta_i\cdot dZ. \end{aligned} \tag{3.8}$$

The solution of the above equation is uniquely given by

$$1+\delta_i L_i(t) = (1+\delta_i L_i(0))\exp\left\{\int_0^t \left(\beta_i\cdot\varphi + \sum_{j=m(s)}^{i}\beta_j\cdot\beta_i - |\beta_i|^2/2\right) ds + \int_0^t \beta_i\cdot dZ\right\}. \tag{3.9}$$

The initial condition (3.4) implies that (3.9) is equal to (3.7). Hence (3.5) is proved.

We may assume that $B_0(0) = 1$, and so $B_1(0) = 1/(1+\delta_0 L_0(0))$ from (3.7). Substituting $i = 1$ into both (3.1) and (3.6), we have

$$\begin{aligned} B_1(T_1) &= B_1(0)\exp\left\{\int_0^{T_1} \bar{\mu}_{m(s)} ds\right\}, \\ &= \frac{1}{1+\delta_0 L_0(0)}\exp\left\{\int_0^{T_1} \frac{\log\{1+\delta_0 L_0(T_0)\}}{\delta_0} ds\right\} = 1. \end{aligned}$$

Inductively it follows that $B_i(T_i) = 1$ for all i. Hence the prices of all bonds are equal to 1 at each maturity date.

Next we shall show that B is arbitrage-free. Let $\xi : \Omega \times [0, T] \to \mathbf{R}$ be an Ito process defined by

$$\frac{d\xi}{\xi} = -\bar{\mu} dt - \varphi \cdot dZ, \tag{3.10}$$

with $\xi(0) = 1$. There exists a unique positive solution

$$\xi(t) = \exp\left\{-\int_0^t (\bar{\mu} + \frac{|\varphi|^2}{2}) ds - \int_0^t \varphi\cdot dZ\right\}. \tag{3.11}$$

Combining (3.6) and (3.11) we have

$$\begin{aligned}\xi B_i &= B_i(0)\exp\left\{\int_0^t\left(-\frac{|\varphi|^2}{2} - \sum_{j=m(s)}^{i-1}\beta_j\cdot\varphi - \frac{|\sum_{j=m(s)}^{i-1}\beta_j|^2}{2}\right) ds - \int_0^t (\varphi + \sum_{j=m(s)}^{i-1}\beta_j)\cdot dZ\right\} \\ &= B_i(0)\exp\left\{-\int_0^t \frac{|\varphi + \sum_{j=m(s)}^{i-1}\beta_j|^2}{2} ds - \int_0^t (\varphi + \sum_{j=m(s)}^{i-1}\beta_j)\cdot dZ\right\} \end{aligned} \tag{3.12}$$

for all i. Hence ξB is a **P**-martingale, and thus B is arbitrage-free.

From (3.3) and (3.10) it is easy to see that the relation of (2.2) holds, and therefore φ is a market price of risk. This completes the proof.

In particular, (3.3) implies

$$\frac{dB_i}{B_i} = \bar{\mu} dt, \qquad t \in (T_{i-1}, T_i]. \tag{3.13}$$

In other words, B_i has null volatility during the last period $(T_{i-1}, T_i]$ before its maturity. This property is essentially the same as the assumption on bond price volatility in the BGM model.

Note that ξ is the state price deflator, and (3.10) shows that $\bar{\mu}$ represents the implied instantaneous interest rate. It is well known that the option price is independent of the choice of the measure. Indeed, let C_{T_i} be a payoff at T_i of a European option. Then the price of this option at t is given by

$$C_t = \frac{1}{\xi(t)} E_t^{\mathbf{P}}[\xi(T_i) C_{T_i}] = B^*(t) E_t^{\mathbf{P}^*}\left[\frac{C_{T_i}}{B^*(T_i)}\right], \tag{3.14}$$

where we denote the conditional expectation under $\mathbf{P}$ and $\mathbf{P}^*$ by $E_t^{\mathbf{P}}$ and $E_t^{\mathbf{P}^*}$, respectively.

Remark The price volatility σ_i of B_i in (3.3), which is given by $-\sum_{j=m(t)}^{i-1} \beta_j$, corresponds to that of the BGM model ([1], (2.5)).

4 Relation to other models

In this section, we explore the relation between the LMRW and models under other measures. In Section 4.1, we induce an LMSP from the LMRW, and in Section 4.2 we consider the model under the forward measure.

4.1 Spot LIBOR measure

We assume that we already have L_i and B_i of the LMRW, which are given by Proposition3.1. We define B^* on $[0, T]$ by

$$B^* = \frac{B_{m(t)}(t)}{B_1(0)} \prod_{j=1}^{m(t)-1} (1 + \delta_j L_j(T_j)),$$

according to (2.6). We already know that B^* is realized by an SFTS θ^* given in (2.5). B^* satisfies (2.12) at each T_i since $B_{m(T_i)}(T_i) = B_i(T_i) = 1$. Obviously, $B^*(0) = 1$ and $B^* > 0$, then we may take B^* as a numeraire. It holds from (3.13) that

$$dB^*/B^* = dB_{m(t)}/B_{m(t)} = \bar{\mu} dt. \tag{4.1}$$

Note that this is a special case of (2.7) where $\sigma_{m(t)} = 0$. Since B^* is continuous and piecewise differentiable, B^* is expressed by

$$B^* = \exp\left\{\int_0^t \bar{\mu} ds\right\}.$$

We see that the above B^* is equivalent to the process (2.13) obtained by log linear interpolation. It follows from (3.11) that

$$\xi B^* = \exp\left\{ -\int_0^t \frac{|\varphi|^2}{2} ds - \int_0^t \varphi \cdot dZ \right\},$$

then ξB^* is a $\mathbf{P}$-martingale.

The B^* numeraire measure (the spot LIBOR measure) $\mathbf{P}^*$ is given by $d\mathbf{P}^*/d\mathbf{P} = \xi(T)B^*(T)$. Since ξB^* is a $\mathbf{P}$-martingale, it follows $\xi B^* = E_t[d\mathbf{P}^*/d\mathbf{P}]$. Moreover, it holds that $d\langle Z,\ \log(\xi B^*)\rangle = -\varphi$. Suppose $Z^* \equiv Z + \int_0^t \varphi dt$, then Z^* is a $\mathbf{P}^*$-Brownian motion from Proposition 2.2. Substituting Z^* into (3.2) and (3.3), we obtain

$$\frac{dL_i}{L_i} = \lambda_i \cdot \sum_{j=m(t)}^{i} \beta_j dt + \lambda_i \cdot dZ^*, \tag{4.2}$$

$$\frac{dB_i}{B_i} = \bar{\mu}_{m(t)} dt - \sum_{j=m(t)}^{i-1} \beta_j \cdot dZ^*. \tag{4.3}$$

(4.2) is equivalent to the LIBOR process (2.11) of the LMSP. Hence the LMSP is implied from the LMRW by the change of numeraire.

In particular, if we set $\varphi \equiv 0$, then $Z^* = Z$. From (3.11),

$$\xi = \exp\{-\int_0^t \bar{\mu}_{m(s)} ds\} = 1/B^*.$$

Hence, $d\mathbf{P}^*/d\mathbf{P} = \xi(T)B^*(T) \equiv 1$, and then $\mathbf{P}^* = \mathbf{P}$ a.e. Consequently, the case where $\varphi \equiv 0$ is also equivalent to the LMSP.

Remark Section 2.2 shows that the LMSP exists non-uniquely. The above LMSP induced from the LMRW is the most trivial case of the LMSP.

4.2 Forward measure

Let $l > 0$ be a positive integer with $l \leq n$. From (3.3) the price process of B_l is given by

$$\frac{dB_l}{B_l} = \left\{ \bar{\mu}_{m(t)} - \sum_{j=m(t)}^{l-1} \beta_j \cdot \varphi \right\} dt - \sum_{j=m(t)}^{l-1} \beta_j \cdot dZ.$$

The B_l numeraire measure $\mathbf{P}_l$ is defined by $d\mathbf{P}_l/d\mathbf{P} = \xi(T)B_l(T)/B_l(0)$. $\mathbf{P}_l$ is called the forward measure, in contrast to the spot LIBOR measure.

From (3.12) it follows that

$$\xi B_l = B_l(0) \exp\left\{ -\int_0^t \frac{|\varphi + \sum_{j=m(s)}^{l-1} \beta_j|^2}{2} ds - \int_0^t (\varphi + \sum_{j=m(s)}^{l-1} \beta_j) \cdot dZ \right\}. \tag{4.4}$$

We see that

$$d\langle Z,\ \log(\xi B_l)\rangle = -\varphi - \sum_{j=m(s)}^{l-1} \beta_j.$$

Because ξB_l is a $\mathbf{P}$-martingale, $\xi B_l = E_t[d\mathbf{P}_l/d\mathbf{P}]$. Suppose

$$Z^l \equiv Z + \int_0^t (\varphi + \sum_{j=m(s)}^{l-1} \beta_j)dt,$$

then Z^l is a $\mathbf{P}_l$-Brownian motion from Proposition 2.2. Substituting Z^l into (3.2), we obtain, for i with $i < l$,

$$\frac{dL_i}{L_i} = -\lambda_i \cdot \sum_{j=i+1}^{l-1} \beta_j dt + \lambda_i \cdot dZ^l, \tag{4.5}$$

$$\frac{dB_i}{B_i} = \left\{ \bar{\mu}_{m(t)} + \sum_{j=m(t)}^{i-1} \beta_j \cdot \sum_{k=m(t)}^{l-1} \beta_k \right\} dt - \sum_{j=m(t)}^{i-1} \beta_j \cdot dZ^l.$$

Thus L and B are expressed under $\mathbf{P}_l$.

5 Theory of real-world simulation

5.1 Market price of risk

This section estimates the market price of risk φ from historical data. In the following we assume that the market is complete and arbitrage-free, meaning the market price of risk is determined uniquely. The solution of (3.2) is expressed by

$$L_i(t) = L_i(0) \exp \left\{ \int_0^t \left(\lambda_i \cdot \sum_{j=m(t)}^{i} \beta_j + \lambda_i \cdot \varphi - |\lambda|^2/2 \right) dt + \int_0^t \lambda_i \cdot dZ \right\}. \tag{5.1}$$

Although L_i was defined for $i = 1, \cdots, n-1$ in the previous section, for simplicity we assume that L_i are defined for $i = 1, \cdots, n$. Let $\{\lambda_i \lambda_j\}$ denote a matrix with components $\lambda_i \lambda_j$. Then the rank of $\{\lambda_i \lambda_j\}$ should be n because of the complete market. For practical use in real-world simulations, we assume that λ_i is an $n \times d$ matrix with $0 < d \leq n$, and Z is a d-dimensional Brownian motion.

Let $\{t_k\}_{k=1,\cdots,J+1}$ be a sequence of observed times with $t_{k+1} - t_k = \Delta t$, where $J+1$ with $J > 0$ is the number of observed times and $\Delta t > 0$ is a constant. Since $\lambda_i \lambda_j = d\langle \log L_i, \log L_j \rangle$, it follows approximately that

$$\lambda_i \lambda_j \Delta t = Cov(\log L_i(t + \Delta t) - \log L_i(t),\ \log L_j(t + \Delta t) - \log L_j(t)),$$

where $Cov(\ ,\ )$ denotes a sample covariance. The Euler integral of (5.1) implies

$$\log L_i(t + \Delta t) = \log L_i(t) + \left\{ \lambda_i \cdot \sum_{j=m(t)}^{i} \beta_j(t) + \lambda_i \cdot \varphi - |\lambda_i|^2/2 \right\} \Delta t + \sqrt{\Delta t} \lambda_i \cdot Z(1), \tag{5.2}$$

where $Z(1) = \int_0^1 dZ$.

In the following, we assume that $T_i = \delta i$ for a fixed positive constant δ. We denote by $K(t, T)$ the implied forward LIBOR on the period $[t + T, t + T + \delta]$ observed at time t. Obviously $L_i(t) = K(t, \delta i - t)$, and we usually observe $K(t, \delta i)$ from the market rather than L_i. It is furthermore not feasible to observe L_i at t_k for all k from the yield curve. We assume that the volatility of $K(t_k - t, \delta i - t)$ is given by $\lambda_i(0)$ on $t \in [0, \Delta t]$ for all k, and set

$$\kappa_j(t_k) = \frac{\lambda_j(0)\delta K(t_k, \delta j)}{1 + \delta K(t_k, \delta j)}$$

for all t_k. Then (5.2) is modified to

$$\begin{aligned} \log \quad K \quad & (t_k + \Delta t, \delta i - \Delta t) = \log K(t_k, \delta i) \\ + \quad & \left\{ \lambda_i(0) \cdot \sum_{j=1}^{i} \kappa_j(t_k) + \lambda_i(0) \cdot \varphi - |\lambda_i(0)|^2/2 \right\} \Delta t + \sqrt{\Delta t} \lambda_i(0) \cdot Z(1), \end{aligned} \tag{5.3}$$

where we set $m(t_k) = 1$ in (5.3) for all k regarding[1] $t_k = 0 + 0$. For convenience, we denote $\lambda_i(0)$ by λ_i below.

Next, simple regression analysis is used to estimate the market price of risk φ. Precisely speaking, φ is chosen to minimize the sum of squared difference between both sides of (5.3), neglecting the random part. Let $\epsilon(\varphi)$ denote the sum of the time-series and cross sections such that

$$\epsilon(\varphi) \quad = \quad \frac{1}{J} \sum_{k=1}^{J} \sum_{i=1}^{n} \left(\Delta K_i(t_k) - \left\{ \lambda_i \cdot \sum_{j=1}^{i} \kappa_j(t_k) + \lambda_i \cdot \varphi - |\lambda_i|^2/2 \right\} \Delta t \right)^2, \tag{5.4}$$

where we set

$$\Delta K_i(t_k) = \log K(t_{k+1}, \delta i - \Delta t) - \log K(t_k, \delta i). \tag{5.5}$$

We here recall some fundamental results of the principal component analysis on the covariance matrix $\{\lambda_i \lambda_j\}$. To avoid confusion, i and l express subscripts for the maturity date T_i and the Brownian motion Z_l, respectively. Obviously $\{\lambda_i \lambda_j\}$ is decomposed into $\lambda_i \lambda_j = \sum_{l=1}^{d} e_i^l \rho_l^2 e_j^l$, where ρ_l^2 is the l-th eigenvalue and $e^l = (e_1^l, \cdots, e_n^l)$ is the l-th eigenvector with $e_1^l > 0$ such that

$$\lambda_i^l = \rho_l e_i^l, \qquad l = 1, \cdots, d. \tag{5.6}$$

The l-th eigenvector e^l can be regarded as the l-th principal component of the covariance matrix. We call $\lambda^l = (\lambda_1^l, \cdots, \lambda_n^l)$ the *l-th volatility.* It holds that $\rho_l \neq 0$ for all $l = 1, \cdots, d$, since the rank of $\{\lambda_i \lambda_j\}$ is equal to d. e^l are furthermore orthonormal to each other, such that

$$\sum_{i=1}^{n} e_i^l e_i^h = \delta_{lh}, \qquad 1 \leq l, h, \leq d, \tag{5.7}$$

where δ_{lh} is the Kronecker delta.

Returning to our subject, since $\lambda_i \cdot \varphi = \sum_{l=1}^{d} \lambda_i^l \varphi_l$ it follows that

$$\frac{\partial \lambda_i \cdot \varphi}{\partial \varphi_l} = \lambda_i^l. \tag{5.8}$$

[1] Obviously if $t_k = 0$ then it follows that $m(t) = 0$. In the above case, we consider the time integral on $(0, \Delta t]$, then we should set $m(t_k) = 1$ in (5.3).

The partial derivative of $\epsilon(\varphi)$ in φ_l is

$$\begin{aligned}\frac{\partial \epsilon(\varphi)}{\partial \varphi_l} &= \frac{-2}{J}\sum_{k=1}^{J}\sum_{i=1}^{n}\left(\Delta K_i(t_k) - \left\{\lambda_i \cdot \sum_{j=1}^{i}\kappa_j(t_k) + \lambda_i \cdot \varphi - |\lambda_i|^2/2\right\}\Delta t\right)\lambda_i^l \Delta t \\ &= -2\sum_{i=1}^{n}\left(E^H[\Delta K_i] - \left\{\lambda_i \cdot E^H[\sum_{j=1}^{i}\kappa_j] + \lambda_i \cdot \varphi - |\lambda_i|^2/2\right\}\Delta t\right)\lambda_i^l \Delta t \qquad (5.9)\end{aligned}$$

for $l = 1, \cdots, d$, where $E^H[\ ]$ denotes a sample mean under $\mathbf{P}$. A direct calculation implies from (5.6) to (5.9) that

$$\frac{\partial^2 \epsilon(\varphi)}{\partial \varphi_l \partial \varphi_m} = 2\sum_{i=1}^{n}\lambda_i^l \lambda_i^m \Delta t^2 = 2\rho_l \rho_m \delta_{lm} \Delta t^2. \qquad (5.10)$$

Since $\rho_l \neq 0$ for $l,\ 1 \leq l \leq d$, ϵ is strictly convex with respect to φ . Hence the solution φ to the minimizing problem is uniquely determined by solving $\nabla \epsilon(\varphi) = 0$. For a simple expression, a constant vector $\gamma = (\gamma_i, \cdots, \gamma_n)$ is defined by

$$\gamma_i = E^H\left[\frac{\Delta K_i}{\Delta t}\right] - \lambda_i \cdot E^H\left[\sum_{j=1}^{i}\kappa_j\right] + |\lambda_i|^2/2. \qquad (5.11)$$

Note that γ is completely determined from the historical data. Substituting (5.11) into (5.9), $\nabla \epsilon(\varphi) = 0$ is reduced to

$$\sum_{i=1}^{n}(\lambda_i \cdot \varphi - \gamma_i)\lambda_i^l = 0, \qquad l = 1, \cdots, d. \qquad (5.12)$$

From (5.6) (5.7) we have the following.

$$\sum_{i=1}^{n}\lambda_i \cdot \varphi \lambda_i^l = \sum_{i=1}^{n}\lambda_i^l \sum_{h=1}^{d}\lambda_i^h \varphi_h = \sum_{h=1}^{d}\rho_l \rho_h \varphi_h \delta_{lh} = (\rho_l)^2 \varphi_l$$

$$\sum_{i=1}^{n}\gamma_i \lambda_i^l = \rho_l \sum_{i=1}^{n}\gamma_i e_i^l$$

Then (5.12) becomes

$$\rho_l \varphi_l = \sum_{i=1}^{n}\gamma_i e_i^l\ . \qquad (5.13)$$

Since the right side is the $l-$th principal component score of γ, we set

$$\zeta_l = \sum_{i=1}^{n}\gamma_i e_i^l. \qquad (5.14)$$

$\zeta = (\zeta_1, \cdots, \zeta_d)$ is a projected image of γ to the principal component space. We call ζ_l the *l-th MPR score*. Consequently, we have the following proposition from (5.13) and (5.14).

Proposition 5.1 *Suppose that the market price of risk φ is given by the solution that minimizes $\epsilon(\varphi)$ of (5.4). Then the market price of risk φ is given by*

$$\varphi_l = \zeta_l / \rho_l, \qquad l = 1, \cdots, d. \qquad (5.15)$$

The norm of the l-th volatility λ^l is given from (5.6) as

$$\| \lambda^l \| = \left\{ \sum_{i=1}^{n} (\rho_l e_i^l)^2 \right\}^{1/2} = \rho_l.$$

Then ρ_l expresses the norm of the l-th volatility. We call ρ_l the *l-th volatility risk.* Consequent from the proposition above, *the market price of risk is explained as the MPR score per unit risk of the volatility.* Radically put, the market price of risk means something like a Sharpe ratio, so the MPR score may correspond to a return to volatility. This interpretation shall be clarified in Section 6.3 in connection with changes in the historical forward LIBOR curve.

5.2 Fundamental properties of real-world simulations

This section investigates a function of the market price of risk in real-world simulations. For this purpose, the rank of $\{\lambda_i \lambda_j\}$ in (5.2) is assumed to be n, which is called a *full-factor model.* Let a constant $\Delta s > 0$ be fixed, and let λ_i and γ_i be constant on $[0, \Delta s]$.

From (5.6) $\lambda^l, \ l = 1, \cdots, n$ are orthogonal to each other, and span $\mathbf{R^n}$. Then it follows from (5.12) that

$$\lambda_i \cdot \varphi - \gamma_i = 0 \tag{5.16}$$

for all i. Substituting this into (5.2), we have

$$\log L_i(\Delta s) = \log L_i(0) + \left\{ \lambda_i \cdot \sum_{j=1}^{i} \beta_j(0) + \gamma_i - |\lambda_i|^2/2 \right\} \Delta s + \sqrt{\Delta s} \lambda_i \cdot Z(1) \tag{5.17}$$

Combining (5.11) with (5.17) we have the following proposition.

Proposition 5.2 *In a real-world simulation of a full-factor model, the forward LIBOR at time Δs is given by*

$$\log L_i(\Delta s) = \log L_i(0) + E^H \left[\frac{\Delta K_i}{\Delta t} \right] \Delta s + \lambda_i \cdot \left\{ \sum_{j=1}^{i} \beta_j(0) - E^H \left[\sum_{j=1}^{i} \kappa_j \right] \right\} \Delta s + \sqrt{\Delta s} \lambda_i \cdot Z(1) \tag{5.18}$$

We remark that the market price of risk vanishes in the above.

Furthermore assume that an initial forward LIBOR $L_i(0)$ is equal to the sample mean of the historical forward LIBOR $E^H[K(t_k, \delta_i)]$. This assumption is naturally practical, for example, as seen in Geiger [5] (p.52–53). It approximately holds that $\sum_{j=1}^{i} \beta_j(0) \approx E^H \left[\sum_{j=1}^{i} \kappa_j \right]$. We therefore obtain the following corollary.

Corollary 5.3 *In the real-world simulation of a full-factor model, if an initial forward LIBOR is equal to the sample mean of the historical forward LIBOR, then it holds that*

$$\log L_i(\Delta s) \approx \log L_i(0) + E^H \left[\Delta K_i / \Delta t \right] / \Delta s + \sqrt{\Delta s} \lambda_i \cdot Z(1) \tag{5.19}$$

Since $E^H[\Delta K_i]$ explains historical trends of the forward LIBOR change, this corollary shows that real-world simulations are approximately equivalent to an empirical term structure model whose drift and volatility structure are obtained from historical data.

In particular, if $L(0)$ is significantly higher than the sample mean of the forward LIBOR, then it approximately holds that $\sum_{j=1}^{i} \beta_j(0) > E^H\left[\sum_{j=1}^{i} \kappa_j\right]$. Then, from (5.18) the simulation results in higher LIBOR rates than (5.19), and vice versa. Similarly, if the initial curve $L_i(0)$ is steeper than the sample mean of the forward LIBOR curve, that is, if $L_i(0) < E^H[K(t_k, \delta i)]$ for small i and $L_i(0) > E^H[K(t_k, \delta i)]$ for large i, then, when considering averages, the simulation results in a steeper LIBOR curve than the initial curve.

As mentioned in Section 4.1, if the market price of risk is taken to be zero the model is equivalent to the risk-neutral model. Substituting $\varphi = 0$ into (5.16), we have $\gamma_i = 0$. We have from (5.17) that

$$\log L_i(\Delta s) = \log L_i(0) + \left\{ \lambda_i \cdot \sum_{j=1}^{i} \beta_j(0) - |\lambda_i|^2/2 \right\} \Delta s + \sqrt{\Delta s}\lambda_i \cdot Z(1). \tag{5.20}$$

For convenience, we call the second term in the above a *risk-neutral drift.* Comparing (5.19) with (5.20), we can say that *the market price of risk is roughly a drift converter from the risk-neutral drift to the historical trend.* This new interpretation is implied because the LMRW admits a full-factor model with an arbitrary volatility structure.

6 Numerical analysis of real-world simulations

This section studies the properties of real-world simulations for practical use. Notations from the previous section remain the same.

6.1 Number of factors

For a practical simulation, we first face the question of how many factors are required to describe the dynamics of the forward LIBOR. Traditionally, principal component analysis has provided solutions to this problem. But real-world simulations have specific properties that principal component analysis cannot capture, as described below.

We consider the full-factor model whose LIBOR process is given by (5.1). Let λ^l be the l-th volatility for l, $l = 1, \cdots, n$. For a positive integer $d < n$, a set $\{\lambda^1, \cdots, \lambda^d\}$ is called *the first d volatilities.* If λ and Z in (5.1) are the first d volatilities and the d-dimensional $\mathbf{P}$-Brownian motion respectively, then the term structure is called the *first d-factor model*, or the *d-factor model* for short. Hence our objective is finding a method to determine the number of factors d such that the d-factor model approximates the full-factor model well. For this purpose, we should remark that the values $\lambda_i \cdot E^H\left[\sum_{j=1}^{i} \kappa_j\right]$ and $|\lambda_i|^2/2$ in the d-factor model are affected by the number of factors. Therefore from (5.11) and (5.15), the numerical value of the market price of risk is also affected by the number d, a subtle dependence that nonetheless affects our objective.

Hence we begin our study with the full-factor model. $\log L_i(\Delta s)$ is exactly described from (5.2) by

$$\begin{aligned} \log L_i(\Delta s) &= \log L_i(0) + \left(\sum_{l=1}^{n} \lambda_i^l \sum_{j=1}^{i} \frac{\delta \lambda_j^l L_j(0)}{1+\delta L_j(0)} - \sum_{l=1}^{n} |\lambda_i^l|^2/2 \right) \Delta s \\ &+ \lambda_i \cdot \varphi \Delta s + \sum_{l=1}^{n} \lambda_i^l \sqrt{\Delta s} Z^l(1). \end{aligned} \tag{6.1}$$

To avoid confusion, we denote L_i, φ_l, ζ_l and γ_i in the first d-factor model by $\breve{L}_i$, $\breve{\varphi}_l$, $\breve{\zeta}_l$ and $\breve{\gamma}_i$ respectively. Hence in the d-factor model, $\log \breve{L}_i(\Delta s)$ is given by

$$\begin{aligned} \log \breve{L}_i(\Delta s) &= \log L_i(0) + \left(\sum_{l=1}^{d} \lambda_i^l \sum_{j=1}^{i} \frac{\delta \lambda_j^l L_j(0)}{1+\delta L_j(0)} - \sum_{l=1}^{d} |\lambda_i^l|^2/2 \right) \Delta s \\ &+ \lambda_i \cdot \breve{\varphi} \Delta s + \sum_{l=1}^{d} \lambda_i^l \cdot \sqrt{\Delta s} Z^l(1). \end{aligned} \tag{6.2}$$

An integer $d > 0$ is chosen such that $\sum_{l=1}^{d} (\rho_l)^2 \approx \sum_{l=1}^{n} (\rho_l)^2$, that is, the accumulated contribution rate of the first d principal components is approximately equal to 1. From (5.6) we have

$$\sum_{l=1}^{d} \lambda_i^l \sum_{j=1}^{i} \frac{\delta \lambda_j^l L_j(0)}{1+\delta L_j(0)} - \sum_{l=1}^{d} |\lambda_i^l|^2/2 \approx \sum_{l=1}^{n} \lambda_i^l \sum_{j=1}^{i} \frac{\delta \lambda_j^l L_j(0)}{1+\delta L_j(0)} - \sum_{l=1}^{n} |\lambda_i^l|^2/2.$$

In the sense of convergence in probability, it follows that

$$\sum_{l=1}^{d} \lambda_i^l \cdot \sqrt{\Delta s} Z(1)^l \approx \sum_{l=1}^{n} \lambda_i^l \cdot \sqrt{\Delta s} Z(1)^l.$$

Principal component analysis usually determines the number of factors from the above approximations, but it remains to show when the following approximation holds:

$$\sum_{l=1}^{d} \lambda_i^l \breve{\varphi}_l \approx \sum_{l=1}^{n} \lambda_i^l \varphi_l \tag{6.3}$$

In (5.15), φ_l is expressed with the denominator ρ_l, and $|\rho_l|$ is decreasing in l, where $|\ |$ is the absolute value symbol. Then it should be remarked that $|\varphi_l|$ is not always decreasing in l, hence (6.3) is not implied by principal component analysis information alone. The difference between both sides of (6.3) is estimated as follows.

$$\left| \sum_{l=1}^{n} \lambda_i^l \varphi_l - \sum_{l=1}^{d} \lambda_i^l \breve{\varphi}_l \right| \le \left| \sum_{l=d+1}^{n} \lambda_i^l \varphi_l \right| + \left| \sum_{l=1}^{d} (\lambda_i^l \varphi_l - \lambda_i^l \breve{\varphi}_l) \right| \tag{6.4}$$

From (5.6) and Proposition 5.1 we have that $\lambda_i^l \varphi_l = \rho_l e_i^l \varphi_l = \zeta_l e_i^l$ and $\lambda_i^l \breve{\varphi}_l = \breve{\zeta}_l e_i^l$. Substituting these into the second term of (6.4), we have

$$\left| \sum_{l=1}^{n} \lambda_i^l \varphi_l - \sum_{l=1}^{d} \lambda_i^l \breve{\varphi}_l \right| \le \left| \sum_{l=d+1}^{n} \zeta_l e_i^l \right| + \left| \sum_{l=1}^{d} (\zeta_l - \breve{\zeta}_l) e_i^l \right|.$$

Since d is sufficiently large, $\breve{\gamma}_i$ is approximately equal to γ_i for all i from (5.11). From (5.11) and (5.14) we then see $\breve{\zeta}_l \approx \zeta_l$ for l with $l \leq d$. Consequently, if $|\zeta_l|$ is sufficiently small for all l with $d < l \leq n$, then $\sum_{l=1}^{d} \lambda_i^l \varphi_l$ approximates $\sum_{l=1}^{n} \lambda_i^l \varphi_l$ for all i. Therefore we have the following proposition.

Proposition 6.1 *If a positive integer d is sufficiently large such that the accumulated contribution rate of the first d components is approximately equal to 1, and $|\zeta_l|$ is sufficiently small for all l with $d < l \leq n$, then the first d-factor model approximates the full-factor model well.*

However, it is difficult to obtain ζ_l for all l in the full-factor model. We may simplify this proposition for practical use as follows.

Corollary 6.2 *For sufficiently large $\tilde{n} \leq n$, let $\breve{\zeta}_l$ $l = 1, \cdots \tilde{n}$ denote the MPR scores in the $\tilde{n}$-factor model. If a positive integer d satisfies the condition in Proposition 6.1, then the first d-factor model approximates the full-factor model.*

In the following, the symbol $\breve{}$ is omitted for simplicity. Furthermore Proposition 5.2 and Corollary 5.3 approximately hold in the d-factor model as below.

Corollary 6.3 *If a positive integer d satisfies the condition in Proposition 6.1, then the following approximation holds in the first d-factor model.*

$$\log L_i(\Delta s) \approx \log L_i(0) + E^H \left[\frac{\Delta K_i}{\Delta t}\right] \Delta s + \lambda_i \cdot \left\{ \sum_{j=1}^{i} \beta_j(0) - E^H \left[\sum_{j=1}^{i} \kappa_j \right] \right\} \Delta s + \sqrt{\Delta s}\lambda_i \cdot Z(1)$$

Additionally, if the initial forward LIBOR is equal to the sample mean of the historical forward LIBOR, then it follows that

$$\log L_i(\Delta s) \approx \log L_i(0) + E^H \left[\Delta K_i / \Delta t\right] \Delta s + \sqrt{\Delta s}\lambda_i \cdot Z(1). \tag{6.5}$$

Usually the number of factors is reduced to make simulation practical. In such cases, (6.5) shows that real-world simulations are approximately equal to a historical term structure, as mentioned in Section 5.2.

6.2 Historical trend of LIBOR

To calculate the market price of risk, it is essential to evaluate γ_i of (5.11) as accurately as possible. In particular, the sample data of ΔK_i immediately determines $E^H[\Delta K_i / \Delta t]$ and the volatility λ_i. Hence this section studies a property of $E^H[\Delta K_i / \Delta t]$.

We define two trend types of the LIBOR in the sample period, as follows. Let $t_1, \cdots, t_{J+1}$ be the same as in Section 5.1. We call the following expectation the *i-th observable trend* of the historical forward LIBOR.

$$E^H[\log K(t_k + \Delta t, \delta i) - \log K(t_k, \delta i)]/\Delta t.$$

Obviously, it holds that

$$
\begin{aligned}
E^H[\log K(t_k + \Delta t, \delta i) - \log K(t_k, \delta i)] &= \frac{1}{J}\sum_{k=1}^{J}\{\log K(t_{k+1}, \delta i) - \log K(t_k, \delta i)\} \\
&= \frac{1}{J}\{\log K(t_{J+1}, \delta i) - \log K(t_1, \delta i)\}.
\end{aligned}
$$

Then we have the following proposition, which means that the observable trend is directly observable from the change of the historical data.

Proposition 6.4 *If it holds that $K(t_{J+1}, \delta i) - K(t_1, \delta i) > 0$, then the i-th observable trend is positive. If $K(t_{J+1}, \delta i) - K(t_1, \delta i) < 0$, then the i-th observable trend is negative.*

We call the following expectation the *i-th rolled trend* of the historical forward LIBOR.

$$
E^H[\log K(t_k + \Delta t, \delta k - \Delta t) - \log K(t_i, \delta i)]/\Delta t
$$

Then from (5.5) the i-th rolled trend is equal to $E^H[\Delta K_i/\Delta t]$. Without loss of generality, we may set $t = 0$ in (5.5). It then holds that

$$
\begin{aligned}
\Delta K_i(0) &= \{\log K(\Delta t, T_i - \Delta t) - \log K(\Delta t, T_i)\} \\
&+ \{\log K(\Delta t, T_i) - \log K(0, T_i)\}.
\end{aligned} \tag{6.6}
$$

If the forward LIBOR curve is upward sloping and remains unchanged, then the first term is negative, and the second one vanishes. As a result, $\Delta K_i(0) < 0$. This is known as the *roll-down* of the forward LIBOR, and the rolled trend expresses the roll-down or roll-up.

Next, we explain an intuitive property of the rolled trend, assuming that the i-th observable trend is stable. If the forward LIBOR curve is upward sloping, then the i-th rolled trend is roughly negative, and conversely, if it is downward sloping, the i-th rolled trend is roughly positive.

Since the roll-down (or up) is an individual trend for each $K(t, \delta i)$, $E^H[\Delta K/\Delta t]$ represents a term structure of the rolled trend. Corollary 5.2 explains that $E^H[\Delta K_i/\Delta t]$ accounts for a major part of the drift term. Hence for some i, if the i-th rolled trend is negative (positive), then from (5.2) the real-world simulation implies a tendency for the roll-down (-up) of L_i. In this sense, Corollary 5.3 is paraphrased as *a real-world simulation roundly reproduces the rolled trend for each L_i.*

There are two actual markets in which to observe the term structure of interest rates. One is the bond market, the other the LIBOR/swap market. In the bond market, L_i is implied from B_i/B_{i+1} with (2.4), so ΔK_i is obtained directly. In this market, however, the term structure is usually observed using the yield curve of the bond market. This approach is fundamentally similar to observations in the LIBOR/swap market. Unfortunately, it is not trivial to observe ΔK_i from the LIBOR/swap yield (or the bond yield), as described below.

From (5.5) ΔK_i is again expressed as follows.

$$
\Delta K_i(t) = \log K(t + \Delta t, T_i - \Delta t) - \log K(t, T_i).
$$

Since $K(t+\Delta t, T_i - \Delta t)$ cannot be directly obtained from the LIBOR/swap yield, $\Delta K_i(t)$ is not observed directly. The most convenient method to estimate ΔK_i is to substitute $K(\Delta t, T_i)$ for $K(\Delta t, T_i - \Delta t)$, as is sometimes seen in the literature. This substitution estimates the observable trend but not the rolled trend, however, so the roll-down (or roll-up) effect inherent in the rolled trend may be missed in simulation. Therefore $K(\Delta t, T_i - \Delta t)$ should be estimated as carefully as possible.

Usually, the time step Δt is chosen to be shorter than δ. We therefore estimate $K(\Delta t, T_i - \Delta t)$ by linear interpolation, as follows.

$$K(\Delta t, T_i - \Delta t) = (1 - \frac{\Delta t}{\delta})K(\Delta t, T_i) + \frac{\Delta t}{\delta}K(\Delta t, T_{i-1}). \tag{6.7}$$

Consequently this paper uses the following estimation for the numerical calculation in Section 7.

$$E^H\left[\Delta K_i\right] = E^H\left[\log\left\{(1 - \frac{\Delta t}{\delta})K(t_{k+1}, T_i) + \frac{\Delta t}{\delta}K(t_{k+1}, T_{i-1})\right\} - \log K(t_k, T_i)\right] \tag{6.8}$$

6.3 Interpretation of the market price of risk

Here, we study the meaning of the market price of risk for a term structure of the rolled trend. From (5.11), γ_i is described as below.

$$\gamma_i = E^H\left[\frac{\Delta K_i}{\Delta t}\right] - \lambda_i \cdot E^H\left[\sum_{j=1}^{i} \kappa_j\right] + |\lambda_i|^2/2 \tag{6.9}$$

From (2.1) and (3.3), $\sum_{j=1}^{i} \kappa_j$ means the price volatility of a bond, which is experimentally quite smaller than the LIBOR volatility. Then the second term of (6.9) is negligibly small.

In particular, we consider the case where $|\lambda_i|^2/2$ is relatively smaller than $E^H\left[\Delta K_i/\Delta t\right]$. Then $\gamma = (\gamma_1, \cdots, \gamma_n)$ is roughly approximated by the rolled trend as

$$\gamma_i \approx E^H\left[\Delta K_i/\Delta t\right], \tag{6.10}$$

because $\zeta_l = \sum_{i=1}^{n} \gamma_i e_i^l$, $\zeta = (\zeta_1, \cdots, \zeta_n)$ is approximately regarded as coordinates of the rolled trend in the principal component space. As mentioned in Section 5, the principal components e^1, e^2 and e^3 respectively explain the movement of the forward LIBOR curve in terms of level, slope and curvature. Similarly, ζ_1, ζ_2, and ζ_3 represent level, slope and curvature factors of the rolled trend. In this context, the l-th market price of risk φ_l *roughly means the l-th component of the rolled trend per the l-th volatility risk.*

Example 6.3.1 Let us assume that the dynamics of the forward LIBOR is expressed by a one-factor model with low volatility. If the slope of the forward LIBOR curve is always steep and the observable trend is stable, then it is likely that the curve almost rolls down, that is, the rolled trend is negative. Hence ζ_1 is negative, and $\varphi_1 = \zeta_1/\sigma$ has a large negative value, since σ_1 is small.

Conversely, if the slope of the curve is almost flat and the level of the curve fluctuates violently, then it is unlikely that the curve rolls down (or up), namely ζ_1 is near zero and σ_1 is large. Then

φ_1 is near zero. If a trader buys all bonds $B_1, \cdots, B_n$, and expects to benefit from the roll-down, then σ_1 is the risk for excess profit. $\Box$

This example shows that *the first market price of risk* φ_1 *is almost a risk-adjusted measure for the roll-down return of the whole forward LIBOR curve.*

Example 6.3.2 We consider the two-factor model case where the forward LIBOR curve is always steep, the whole observable trend is steepening, and the volatility is relatively low. Then ζ_1 and ζ_2 are almost negative, and accordingly φ_1 and φ_2 are negative. If a trader buys a short-maturity bond and sells a long-maturity bond, then the expected benefit is from the steepening of the curve. Therefore the magnitude of the second volatility σ_2 represents positional risk. The second market price of risk $\varphi_2 = \zeta_2/\sigma_2$ explains certainty of the steepening ζ_2. $\Box$

We may say that $\varphi_2 = \zeta_2/\sigma_2$ means *almost a risk-adjusted measure of steepening position.* Furthermore φ_3 is a risk-adjusted measure of the butterfly trading strategy, which expects to benefit from curvature change. Accordingly φ_4 and others can be interpreted in a similar way.

Thus the market price of risk can be roughly estimated for various cases by arguments analogous to the above. Table 6.1 shows rough estimates of the first market price of risk φ_1 in each case. For example, the first case in Example 6.3.1 corresponds to the case of "upward sloping" and "stable" in the table, which is "negative" (that is, φ_1 is almost negative). Table 6.2 shows rough estimates of φ_1 and φ_2 in connection with the two factors of the observable trend, where we assume that almost forward LIBOR curves in sample data are upward sloping. Example 6.3.2 corresponds to the case of "steep" in the table, where φ_1 and φ_2 are estimated as "negative".

In contrast, recall the original definition of the market price of risk. In the LMRW, Proposition 3.1 and (2.1) implies that $\mu_i(t) = \bar{\mu}_{m(t)} - \sigma_i \cdot \varphi$, where $\sigma_i = \sum_{j=m(t)}^{i-1} \beta_j$ is a price volatility of B_i. In the one-factor model, it follows that

$$\varphi = -(\mu_i(t) - \bar{\mu})/\sigma_i, \tag{6.11}$$

where $\mu_i - \bar{\mu}$ is regarded as a risk premium of the i-th bond in the traditional sense. It usually holds that $\mu_i > \bar{\mu}$, and the price volatility σ_i is a risk factor of the bond B_i. In this context, the market price of risk has been interpreted as a risk premium per unit risk. Furthermore, there are sometimes attempts to explain the market price of risk using state variables such as the instantaneous spot interest rate r, which corresponds to the spot LIBOR $L_i(T_i)$ in the LIBOR market model. Indeed, the functional form $\varphi(r) = \varphi r$ is assumed in the model of Cox et al. [2]. Stanton [14] empirically estimates the market price of risk according to the form of $\varphi(r)$. In contrast, our results explain the market price of risk according to changes in the historical forward LIBOR curve, rather than the state of the curve.

6.4 Procedures for real-world simulations

This section summarizes a procedure for simulation in the LMRW according to the results of the previous sections. For simplicity, $\delta = T_{i+1} - T_i$ is assumed to be constant. For a constant time step $\Delta t > 0$, let $t_j,\ j = 1, \cdots, J+1$ be a sequence of observing days, where $J+1$ is the number of days. $K(t_j, \delta i),\ j = 1, \cdots, J+1$ for all i is the implied forward LIBOR observed at t_k. Assuming that the volatility λ and the market prices of risk are constant over the period $[0, \delta]$, the procedure is as follows:

1) $K(t_j + \Delta t, \delta i - \Delta t)$ is obtained by interpolation of (6.7), and $\Delta K_i(t_j)$ is given by (5.5) for all $i, j,\ i = 1, \cdots, n,\ j = 1, \cdots, J$.

2) The eigenvalues $(\rho_l)^2$, the eigenvectors e^l, and the volatilities λ^l are obtained from principal component analysis on $Cov(\Delta K_i,\ \Delta K_j)/\Delta t$.

3) $E^H\left[\Delta K_i\right]$ is obtained from the sample data for all i.

4) For a sufficiently large integer $\tilde{n}$ with $\tilde{n} \leq n$, we calculate $E^H\left[\sum_{j=1}^{i} \kappa_j\right]$ in the $\tilde{n}$-factor model from the sample. Then $\gamma_i,\ i = 1, \cdots, \tilde{n}$ are obtained by (5.11).

5) The MPR score ζ_l is obtained from (5.14) for $l,\ l = 1, \cdots, \tilde{n}$.

6) Examining the accumulated contribution rate and the MPR score, the number of factors $d,\ d \leq \tilde{n}$ is decided by Corollary 6.2.

7) From Proposition 5.1 the market price of risk $\varphi_l,\ l = 1, \cdots, d.$ are obtained from ζ_l in the fifth step.

8) We set the initial LIBOR $L_i(0)$, and generate an d-dimensional sequence of standard normal random numbers.

9)$L_i(\delta)$ is simulated from (5.1) as

$$L_i(\delta) = L_i(0) \exp\left\{ (\lambda_i \cdot \sum_{j=1}^{i} \kappa_j + \lambda_i \cdot \varphi - |\lambda|^2/2)\delta + \sqrt{\delta}\lambda_i \cdot Z(1) \right\}, \tag{6.12}$$

where $Z(1)$ corresponds to the sequence of standard normal random numbers.

10) The next step of the simulation is executed by repeating 9).

We remark that the first and sixth steps characterize the simulation in the LMRW. Additionally, Table 6.1 and 6.2 help to examine the values of the market price of risk for observable trends in the sample data.

7 Numerical examples

7.1 Sample data and number of factors

To show numerical examples of the real-world simulation, we obtained the Japanese LIBOR swap rates for April 2007 through August 2009, and used the cubic spline algorithm to interpolate the interest rates at semiannual intervals.We set $\delta = 0.5$ (year), and solved for 6-month forward LIBOR by bootstrapping the interpolated interest rates. Figure 7.1 shows the forward LIBOR at 0, 2, 5, and 10 years from 2 Apr 2007 to 31 Aug 2009, divided into two periods. The earlier period A covers 2 Apr 2007 to 16 Jun 2008, where the forward LIBOR shows a tendency to rise. The later period B covers 16 Jun 2008 to 31 Aug 2009, where the forward LIBOR shows a tendency to fall. Figure 7.2 presents the forward LIBOR curves at three days (2 Apr 2007, 16 Jun 2008 and 31 Aug 2009). From Proposition 6.4, the observable trend in period A is bear-flattening, and that in period B is bull-steepening.

We set $\Delta t = 0.08$ (20 days). According to the procedure in Section 6.4, the covariance matrix $Cov(\Delta K_i, \Delta K_j)$ is estimated from the sample. Table 7.1 shows the eigenvalues $|\rho_l|^2$, contribution rates, the MPR score ζ_l, and the market prices of risk φ_l for each component, where ζ and φ are calculated using the 8-factor model. The first three components explain more than 98 percent of the covariance. Both periods A and B show that the market price of risk is not always decreasing in l as mentioned in Section 6.1.

Table 7.1 shows that $\varphi_1 = 0.038$ and $\varphi_2 = 0.758$ in period A. Since the observable trend is bear-flattening in this period, these values are explained by Table 6.2, where φ_1 is near zero and φ_2 is positive in the "bear flat" column. The MPR score ζ_l is relatively small for $l > 3$. Hence the first three-factor model may be sufficient for a simulation using the data of period A.

Period B in Table 7.1 shows that $\varphi_1 = -1.094$ and $\varphi_2 = -1.718$. Since the observable trend in this period is bull-steepening, these results are also explained by Table 6.2, where φ_1 and φ_2 are negative in the "bull steep" column. In particular, the fourth market price of risk shows the largest value, and ζ_l is relatively small for $l > 6$. Hence the first four- or five-factor model may be required for simulations using the data of period B. This is a trivial case in that the number of factors is determined by the MPR score, rather than by principal component analysis.

Figure 7.3 shows the first four eigenvectors $e^l,\ l = 1, \cdots, 4$ in periods A and B, identified with the first four principal components. As shown by Litterman and Scheinkman [7], the first three components explain the changes in "level", "slope", and "curvature" of the forward LIBOR curve, respectively. In both periods, the fourth component seems to affect the movement of the short-term forward rates. This is similar to the observations in several studies, for example, Longstaff et al. [8].

Figure 7.4 presents γ and three terms of (6.9), where $E[\Delta K]$, $\lambda E[\sum \kappa]$ and $\lambda/2$ are abbreviations of $E^H\left[\Delta K_i/\Delta t\right]$, $\lambda_i \cdot E^H\left[\sum_{j=1}^{i} \kappa_j\right]$, and $|\lambda_i|^2/2$ respectively. We see that $\lambda_i \cdot E^H\left[\sum_{j=1}^{i} \kappa_j\right]$ is negligibly small in both cases. In period A, $|\lambda_i|^2/2$ is not smaller than $E^H\left[\Delta K_i/\Delta t\right]$. Hence the approximation (6.10) does not hold for this case. On the other hand, in period B $|\lambda_i|^2/2$ is relatively smaller than $E^H\left[\Delta K_i/\Delta t\right]$, so the approximation (6.10) holds. Period B is expected to show characteristics of the real-world simulation better than period A. The next section presents some numerical examples of the real-world simulation, focusing on period B.

7.2 Simulations in LMRW and LMSP

We now consider the following four cases, using the data of period B. Case B1 is a four-factor real-world simulation according to the procedure in Section 6.4. Case B2 is a four-factor simulation under the spot LIBOR measure, executed by setting $\varphi = 0$ in Case B1. Case B3 is a three-factor real-world simulation based on Case B1. Case B4 is a four-factor simulation under the real-world measure such that the substitution of $K(t_{k+1}, \delta i)$ for $K(t_{k+1}, \delta i - \Delta t)$ is used to calculate $\Delta K_i(t_k)$ of (5.5). In other words, ΔK is calculated by the observable trend instead of the rolled trend. Table 7.2 summarizes the four cases.

The implied forward LIBOR observed on 31 August 2009 is taken as initial rates, as presented in Figure 7.2. In each case, $L_i(\delta)$ is calculated using a single-period simulation. We examine the four cases by comparing the mean of each $L_i(\delta)$, and evaluate $E[\log L_i(\delta)]$ rather than $E[L_i(\delta)]$ for convenience. For given $L(0)$, λ, and φ, (5.2) implies that

$$E[\log L_i(\delta)] = \log L_i(0) + \left\{ \lambda_i \cdot \sum_{j=1}^{i} \kappa_j(0) + \lambda_i \cdot \varphi - |\lambda_i|^2/2 \right\} \delta. \tag{7.1}$$

When $\varphi = 0$, an LMRW is equivalent to an LMSP and $\mathbf{P} = \mathbf{P}^*$ holds. Then $E[\ ] = E^*[\ ]$ holds for Case B2, where $E^*[\ ]$ denotes the expectation under $\mathbf{P}^*$. Then (7.1) is also available for Case B2, and accordingly $E[\log L_i(\delta)]$ is obtained for all cases by direct calculation of (7.1). Figure 7.5 presents the initial forward LIBOR and $E^H[K(t_k, \delta i)]$ of Period B. Figure 7.6 compares the mean $E[\log L_i(\delta)]$ to the initial forward LIBOR in log scale for each case.

To clarify the results of Case B1 we first examine Case B2, a risk-neutral simulation in the LMSP. In a risk-neutral simulation, the mean of the forward LIBOR curve at $t = \delta$ is close to the parallel change of the initial curve to the left for δ. In other words, the shape of the curve of $E[L_i(\delta)]$ is roughly close to $L_i(0)$ for all i. Naturally Case B2 of Figure 7.6 shows this property, so $E[\log L_i(\delta)]$ is upward sloping at the short end. We furthermore note that Case B2 has a bear-flattening tendency.

Next we examine Case B1. Recall that the observable trend is bull-steepening in Period B, and the initial forward LIBOR is steeper than the mean of the forward LIBOR, as shown in Figure 7.5. Then from Corollary 5.2, the real-world simulation of Case B1 implies a bull-steepening tendency, as confirmed in Figure 7.6. This is quite different from Case B2. In particular, the initial forward LIBOR is locally downward sloping at the short end. It is remarkable that the mean of the simulated forward curve at $t = \delta$ in Case B1 is slightly downward sloping at the short end, in contrast with Case B2.

Case B3 is a three-factor simulation in the LMRW. The results shown in Figure 7.6 are similar to Case B1, except at the short end, where $E[\log L_i(\delta)]$ is almost flat. The results of Case B4 in Figure 7.6 seem to be an intermediate between Case B1 and Case B2. Hence Case B4 avoids the effect of both the fourth factor and the real-world simulation. Accurate simulations are therefore necessary to properly estimate $K(\Delta t, T_i - \Delta t)$.

Consequently both Case B1 and B3 imply plausible results as real-world simulations. In particular only Case B1 implies that the downward slope at the short end will continue for a while. We suggest that the fourth factor may affect the movement of the short term forward LIBOR. Finally, Figure 7.7 shows 50 simulations of the forward LIBOR at $t = \delta$ in Case B1.

8 Conclusion

We have specified the LMRW within the framework of Jamshidian [6]. In the LMRW, we assume a constant yield for the shortest maturity bond so the real-world model is simply specified. At the same time, the explicit form of the state price deflator is shown; thus, the LMRW is applicable not only for real-world simulation but also for option pricing.

Furthermore, under the assumption that the market price of risk is constant, we found an explicit expression for the market price of risk in the framework of the LMRW. This makes it possible to study the fundamental properties of real-world simulations, and gives the following fundamental results related to the market price of risk and real-world simulations.

1) The market price of risk has an approximate function as a drift converter from a risk-neutral drift to a historical trend of the forward LIBOR.

2) Real-world simulations are similar to empirical simulations where drift and volatility are obtained from historical data.

3) The number of factors for real-world simulation is determined not only by principal component analysis, but also by the MPR score.

4) When the volatility is low, the market price of risk is almost determined by the rolled trend. It is therefore important to estimate ΔK_i properly for numerical simulations.

5) The market price of risk is roughly explained by changes in the historical forward LIBOR curve, rather than by the state of the curve.

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Table 6.1 The first market price of risk φ_1 in connection with the observable trend and average shape of the initial forward LIBOR curve

Market price of risk φ_1		Observable trend		
		Fall	Stable	Rise
Shape of forward LIBOR curve	Upward sloping	Negative	Negative	Near zero
	Flat	Negative	Near zero	Positive
	Downward sloping	Near zero	Positive	Positive

Table 6.2 The observable trend the market price of risk
The forward LIBOR curves are upward sloping in the sample

		Observable trend					
		Bull steep	Steep	Bear steep	Bull flat	Flat	Bear flat
Market price of risk	φ_1	Negative	Negative	Near zero	Negative	Negative	Near zero
	φ_2	Negative	Negative	Negative	Positive	Positive	Positive

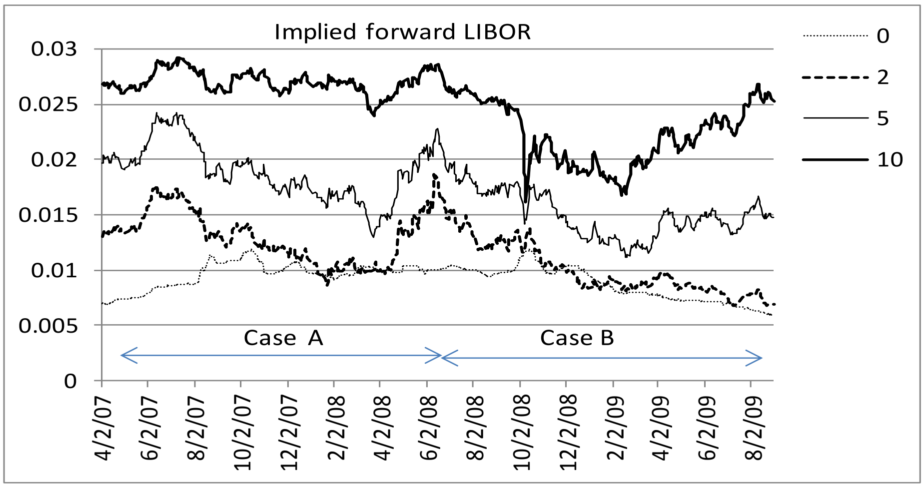

Figure 7.1 Implied forward LIBOR, JPY LIBOR/swap market from 2007/3/27 to 2011/6/9. 6 month forward LIBOR in 0, 2, 5 and 10 years. Swap data are provided by Mizuho Information & Research Institute.

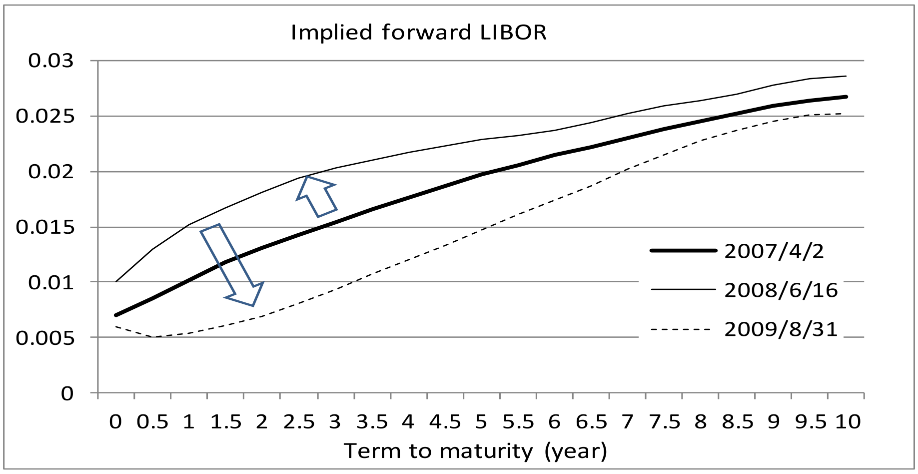

Figure 7.2 Implied forward LIBOR curve

Case A is from 2007/4/2 to 2008/6/16, Case B is from 2008/6/16 to 2009/8/31

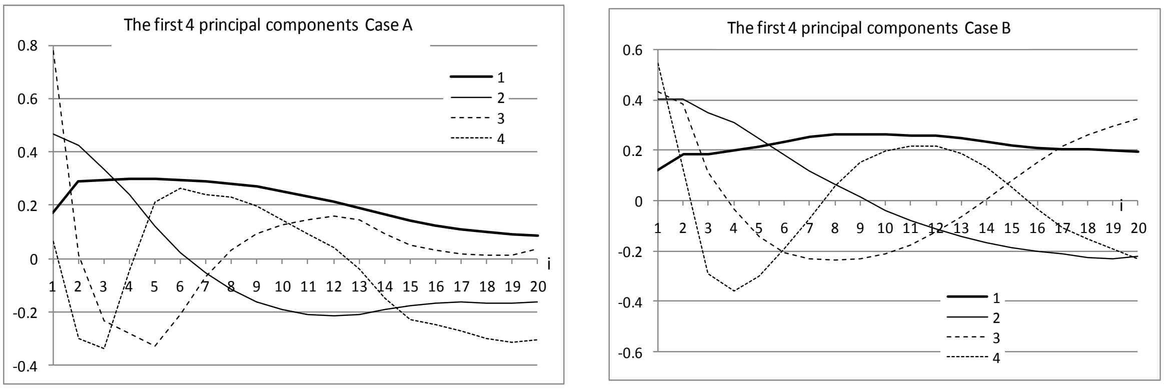

Figure 7.3 The first four principal components for Period A and Period B

Δt=0.08 (20 days)

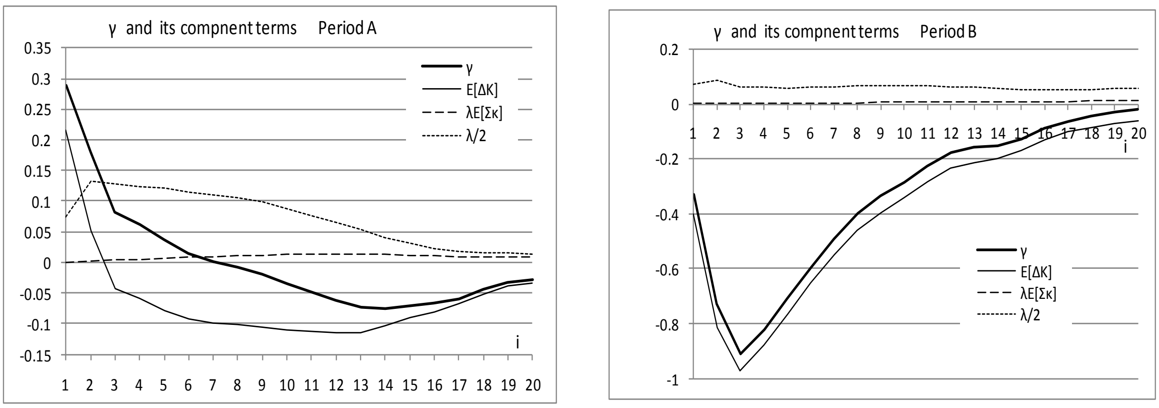

Figure 7. 4 γ and its components for Period A and Period B

Table 7.1 Eigenvalues and the market price of risk, JPY LIBOR/Swap market

Period A 2007/4/3-2008/6/16

	Eigenvalue ρ^2	Contribution rate	Accumulated contribution rate	MPR score ζ	Market price of risk φ
1-st	2.615	0.8999	0.8999	0.061	0.038
2-nd	0.2366	0.0814	0.9813	0.369	0.758
3-rd	0.0284	0.0098	0.9911	0.126	0.746
4-th	0.0146	0.0050	0.9961	0.014	0.114
5-th	0.0059	0.0020	0.9982	0.016	0.205
6-th	0.0024	0.0008	0.9990	0.061	1.226
7-th	0.0011	0.0004	0.9994	-0.034	-1.017
8-th	0.0005	0.0002	0.9996	0.004	0.155

Period B 2008/6/16-2009/8/31

	Eigenvalue ρ^2	Contribution rate	Accumulated contribution rate	MPR score ζ	Market price of risk φ
1-st	1.756	0.7191	0.7191	-1.449	-1.094
2-nd	0.4904	0.2008	0.9199	-1.203	-1.718
3-rd	0.1481	0.0607	0.9805	0.082	0.213
4-th	0.0240	0.0098	0.9904	0.398	2.570
5-th	0.0100	0.0041	0.9945	0.209	2.089
6-th	0.0091	0.0037	0.9982	0.053	0.550
7-th	0.0014	0.0006	0.9988	-0.004	-0.115
8-th	0.0011	0.0005	0.9992	0.071	2.113

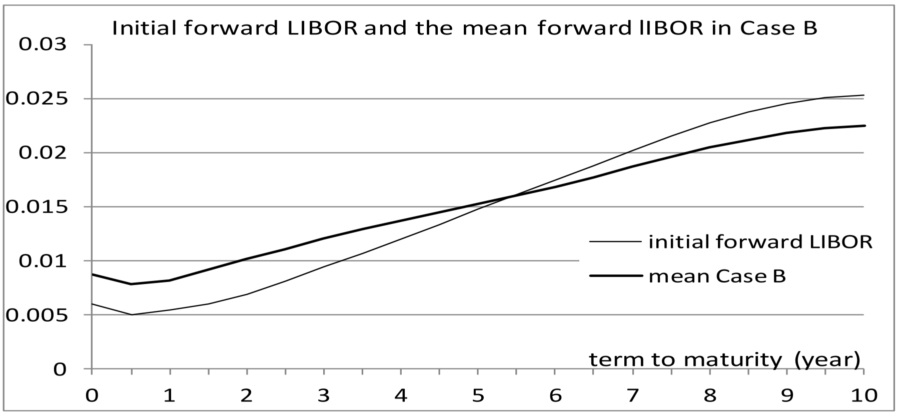

Figure 7.5 Initial forwad LIBOR at 2008/8/31 and the average of the implied forward LIBOR in Case B

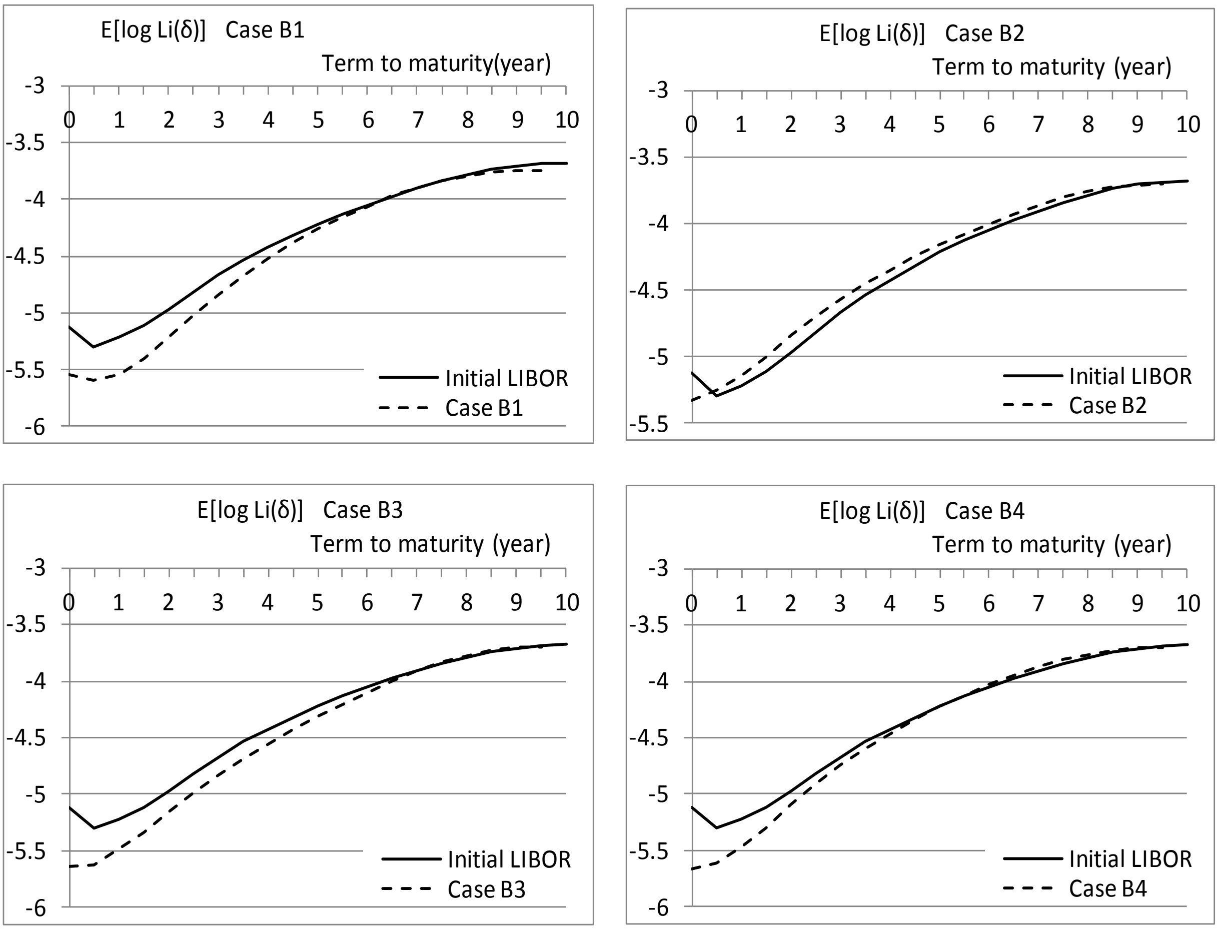

Figure 7.6 Initial forward LIBOR and the mean of log Li(δ) in Case B1, B2, B3, B4

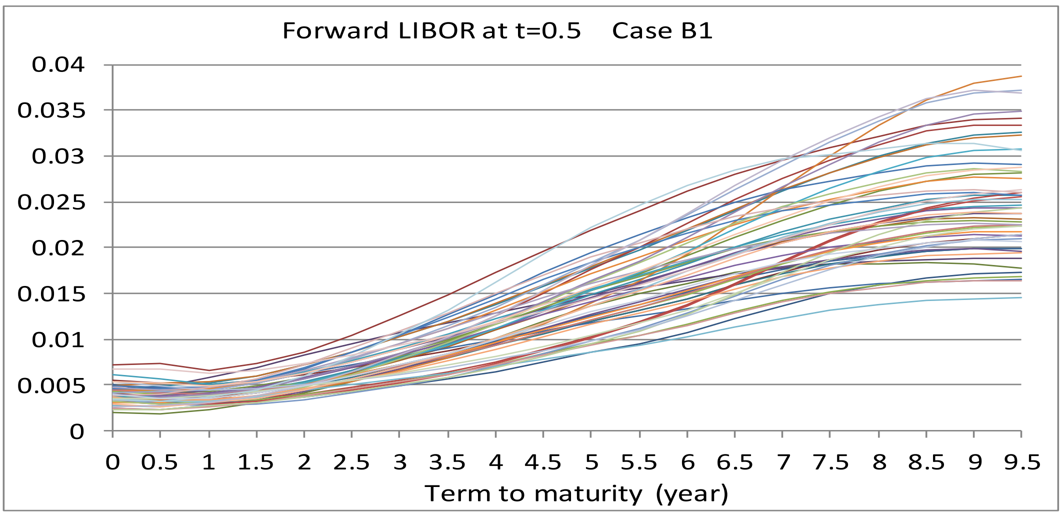

Figure 7.7 Case B1 Forward rates at t=0.5 year 50 simulations

Four-factor real-world simulation